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Influence of hull resiliency on the response of shipboard equipment to shock

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OF SHIPBOARD EQUIPMENT TO SHOCK

by

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Submitted to the Department of Naval Architecture and Marine Engineering on 19 May 1962 in partial fulfillment of the requirements for the Master of Science Degree in Naval Architecture and Marine Engineering and the Professional Degree, Naval Engineer.

ABSTRACT

Current methods of shock design and analysis rely primarily on the input of large amounts of experimental data, assembled from underwater explosion tests, to describe the shock motion inputs to their systems. The object of this investigation is to arrive at a more complete characterization of the shock motion input, a net effect representing the interaction between the water applied pressure loading, the hull structural response and the equipment - foundation response.

The model chosen is the cross section of a simple cylindrical shell with an inboard mounted simple oscillator, immersed in an acoustic medium. The input to the water-shell system is an exponentially decaying plane step wave incident normal to the cylinder longitudinal axis.

The analysis employs the use of four-pole mechanical impedance techniques and Laplace transform methods. An analytic expression is derived, which has the form of a mechanical quadripole impedance. It relates shock wave pressure to equipment response, as a function of shell geometry and oscillator mass and natural frequency. The lack of rigor associated with the description of a distributed mass system as a point phenomenon along with other restrictive conditions, cause the expression to be better termed a "quasi-impedance".

The problem formulation and solution has limitations on both the mathematical and physical side and at this stage the results are to be regarded mainly as exploratory. The "quasi-impedance" is now capable of predicting qualitatively the effects of parameter variation on the response of simple equipments to shock wave inputs. Additional refinements of the shell model and comparison with experimental data will enable a more rigorous evaluation of quantitative effects.

Thesis Supervisor: John R. Baylis

Title: Associate Professor of Naval Engineering

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Credit

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NOMENCLATURE

A	=	cross-sectional area of cylindrical shell
a	=	radius of the cylinder
a_i, b_i ($i=1, n$)	=	coefficients in system characteristic equation
B	=	non-dimensional shock wave decay constant
c	=	sound speed in medium
e_{ij}	=	general notation for four-pole parameters
E	=	Young's modulus of elasticity for shell
F_i	=	point force per unit longitudinal length where i refers to a particular component output or input point
F_{in}	=	modal component of F_i
F'_{in}	=	force per unit shell surface area
$\bar{g}_n$	=	Haywood's "after flow" constant, average value
$G(s)$	=	general form of transfer function
h	=	shell thickness
H_i ($i=1, n$)	=	denominators of partial fraction expansion
I	=	moment of inertia of shell cross section
$I_{n\theta}$, I_n	=	incomplete and complete modified Bessel functions of the first kind of order n, respectively
$I'_{n\theta}$, I'_n	=	first derivative of the function with respect to the argument
k	=	equivalent foundation stiffness per unit longitudinal length
K_1	=	$2\pi a\ L_{eff}$

L_{eff}	=	ratio of the longitudinal length of shell feeling the influence of the equipment reaction to the actual equipment length
m	=	mass of shell per unit area
m_n	=	modal or generalized masses of shell per unit area
M_1	=	mass of equipment model
n	=	mode number and equivalently the number of circumferential waves of mode
P_i	=	free-field shock wave pressure
P_r	=	radiated and diffracted wave fluid pressure
P_{in}, P_{rn}	=	generalized or modal components of P_i, P_r
P_o	=	shock wave peak pressure
q_n, $\dot{q}_n$	=	generalized or modal deflection in the radial direction and its derivative with respect to time, the generalized velocity
$\bar{\dot{Q}}_n$	=	Laplace transform of $\dot{q}_n$
r, θ	=	cylindrical coordinates
s	=	complex frequency operator
s_i $(i=1, n)$	=	roots of characteristic equation
t	=	time
T	=	non-dimensional time = ct/a
$u(t)$	=	step function
U_i	=	free-field shock wave particle velocity
U_r	=	radiated and diffracted wave fluid particle velocity

U_{in}, U_{rn}	=	modal components of U_i, U_r
V_i	=	point velocity where i refers to the particular component output or input point
V_{in}	=	modal component of V_i
V_o	=	peak velocity of quasi-impedance function
w	=	radially inward net velocity of shell as a function of θ
$\tilde{\dot{x}}(s)$, $\dot{x}(t)$	=	relative velocity between equipment mass and shell, in complex frequency and time domain, respectively
Z_i	=	general notation for impedance function
β	=	shock wave decay constant, sec^{-1}
Δ	=	notation for characteristic equation
Γ	=	gamma function
ϵ_n	=	Neumann Factor; 1 for $n = 0$; 2 for $n > 0$
ν	=	Poisson's ratio
$\theta(T)$	=	angle of shock wave envelopment of cylinder $= \cos^{-1}(1 - T)$
ρ	=	mass density of medium
ρ_s	=	mass density of shell material
ω_{mach}	=	circular frequency of equipment foundation model
σ_n	=	non-dimensional stiffness of shell $= a\,\bar{\omega}_n^2/\rho c^2$
ω_n	=	modal circular frequency of shell in-vacuo
$\bar{\omega}_n$	=	shell stiffness in-vacuo

Φ = potential function of diffracted and radiated waves in fluid

General

$\overline{\text{function}}$ = Laplace transform of the function with respect to time

I. INTRODUCTION

A. Shock-Resistant Equipment Design

In recent years routine tests have been conducted by the Navy to measure the resistance of naval ships to underwater shock waves from non-contact underwater explosions. These tests have revealed that incapacitating damage to shipboard equipments has occurred at shock intensities of only a fraction of those required to produce lethal hull damage[27]. At the present time, considerable attention is being devoted by various design analysis methods to the prediction of the responses that can be expected by equipments to typical motion inputs. Because of the complicated interaction problems in the general case of a shock wave-shell-equipment interaction, shock motion inputs have been arrived at only after evaluation of vast amounts of experimental data from realistic underwater explosion tests. Rational utilization of experimental data requires correlation between test charge intensity, shock motion recorded, and shell-equipment interaction parameters, to include relative mass, on-board location, and natural frequencies. By realistic modeling of equipments for which dynamic response data from underwater explosion tests are available, it is possible to solve for the shock motion input at some reference point, i.e., inboard side of shell. A wide range of dynamic situations have to be investigated before a general set of input curves can be arrived at.

An alternate approach is to tackle the shell-shock wave interaction, making sufficient assumptions to make the problem solvable and to investigate:

I. INTRODUCTION

1. Shock-Resistant Equipment Design

In recent years routine tests have been conducted by the Navy to measure the resistance of naval ships to underwater shock waves from non-contact underwater explosions. These tests have revealed that incapacitating damage to shipboard equipment can occur at shock intensities of only a fraction of those required to produce serious hull damage(17). At the present time, considerable attention is being devoted by various design analysis methods to the prediction of the responses that can be expected by equipments to typical shock inputs. Because of the complicated interaction problem in the general case of a shock wave-shell-equipment interaction, shock motion inputs have been arrived at only after evaluation of vast amounts of experimental data from realistic underwater explosion tests. Intelligent utilization of experimental data requires correlation between shock charge intensity, shock motion recorded, and shell-equipment interaction parameters, to include relative mass, equipment location, and natural frequencies. To postulate motion of equipments for which dynamic response data from underwater explosion tests are available, it is possible to solve for the shock motion input at each reference point, i.e., inboard side of shell. A wide range of dynamic situations have to be investigated before a general set of input curves can be arrived at.

An alternate approach is to tackle the shell-shock wave interaction, making sufficient assumptions to make the problem tractable and to investigate:

(1) the qualitative validity of the solution

(2) possibility of the introduction of experimental information to adjust the quantitative solution.

This thesis shall investigate the feasibility of analytically describing the shock wave - hull - equipment interaction for the case of a submarine-type model. This description will take the form of an impedance. Due to non-rigorous, yet physically meaningful, assumptions, the result will be termed a quasi-impedance.

B. Shock Design Inputs to Several Analysis Methods.

A design input may be defined as a description of the shock forcing or motion function to the design system. Dependent on the system chosen, it could be a velocity or acceleration as a function of time at some reference point in the ship or some pressure-time description of the shock-wave itself. A summary of the design inputs to the three best-known design methods are outlined below.

1. Static Shock Design Number

The most rudimentary of the methods, the input is a shock design number, dependent upon equipment size, ship type and the attack direction[42]. It is applied to the equipment weight to produce an equivalent force which the foundation and mounting should withstand. A proposed modification to the approach would modify the number by a resiliency factor based upon the ratio of equipment natural frequency to the dominant frequency of the input forcing function, as determined from peaks in the frequency spectrum of the input motion[41].

2. Normal Mode.

The normal mode approach currently describes an equivalent shock motion input (frequency domain) to a mathematical model at some inboard fixed-base reference point, i.e., for a submarine, the inboard side of the pressure hull. The fundamental description of the input is called the shock spectrum. The shock spectrum is defined as the trace of the maximum absolute response of a single degree of freedom system to the applied rigid base motion as the frequency of the oscillator is varied from 0 - ∞. Its drawback as a design input for naval shock lies in the fact that it fails to account for structural interaction between the oscillator and the foundation exciting it. The spectrum provides no information with regard to time history, but rather an upper bound on the maximum value of the response as a function of frequency. Extension of the non-damped vibration absorber from classical vibrations [7] and discrete-mass laboratory experiments have led some observers[37][38] to propose that the values of interest in a shock spectrum tend to lie in the nulls of the plot, rather than at the peaks. A fiducial envelope of the peaks is valid as an input when dynamic reaction of the driven component is unable to affect the motion of the driving component. So as to account for the interaction, the fundamental shock spectrum is modified by equipment weight, on-board location, and natural frequency. The result is termed a "design shock spectrum" and is the design input[28]. By assembling enough experimental data (equipment response) ranging over a sufficient band of equipment natural frequencies, relative mass ratios between equipment-foundation

and hull mass, on-board location, charge (shock-wave) intensities, and ship types, design inputs can be arrived at. Each point on a design input chart requires equipment-foundation modeling and inverse normal mode solution for the forcing function required to produce the measured response. For the submarine case, a modification to this approach has been suggested which moves the reference point outside the shell and adds hull flexibility to foundation flexibility. Hull resiliency would be experimentally determined(28)(41).

3. Iteration.

Essentially a numerical method, it describes the hull - foundation - equipment interaction by a non-linear two-mass system. It is solved by convergent iteration on the equations of motion. The design input is a pressure-time history of the free-water shock wave which is the forcing function on a discrete mass describing the hull "effective mass"(36).

C. Mechanical Impedance Approach.

Concurrent with renewed interest in the adaptation of classical dynamics to complicated structures, has been the adoption of an analytic tool from electrical engineering known as the impedance technique.

The use of impedance permits combinations of a large complicated arrangement of components to be handled as a single entity. Computational efficiency is increased because only input and output data are handled, eliminating explicit introduction of all the parameters characterizing each component "black box". Further, the parameters contained in each "black box" are a function only of that particular component and are not influenced by the component's

surroundings. The use of such a technique in the analysis of shock motions appears ideal in that it provides:

(1) a flexible procedure allowing for the introduction of empirical and quasi-analytic functions describing "black box" interactions.

(2) direct utilization of experimental data

(3) maximum ease of evaluation of effects on the system response due to component changes, i.e., removal, alteration, replacement.

To characterize the interaction of a "black box" with its surroundings, the impedance technique uses terms such as load or driving point impedance, output impedance, and transfer impedance. A further refinement of the impedance technique formalizes these interaction impedances into two linear equations which relate the input point, 1, to the output point, 2, in terms of their four-pole parameters:

$$F_1 = e_{11} F_2 + e_{12} V_2$$

$$V_1 = e_{21} F_2 + e_{22} V_2$$

This approach is called the Four-Pole Parameter Technique and is an analytic tool in this investigation(1)(23).

D. Shock Wave - Hull Interaction

Pressure wave interaction with cylindrical obstacles was investigated classically by Raleigh(10). The problem has received revived interest in the application of the basic approach to obstacles whose thin cross sections are allowed to deform elastically.

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The earliest of these studies[47] used a modal analysis and retained the full, exact form of the diffracted and radiated wave. The analysis was handled by transform methods. Due to the complexity of the solution (presence of complex poles and branch points), inversion by contour integration for more than several modes was tedious. Mindlin and Bleich[18], by introducing an approximation valid for small time, simplified the solution of[47] and presented numerical results for the first three modes of cylinder response. The simplification amounted to giving the propagation characteristics of a plane wave to the radiated and diffracted wave. Extensional effects in all modes, but the dilatational mode, were neglected. Baron[15] extended the analysis of [18] to include both inextensional and extensional effects in all modes, using the plane wave approximation. More recently, Haywood[20] has contributed a derivation allowing for a more accurate approximation to the characteristic impedance of a cylindrical wave allowing for an "after-flow" effect. The approximation reduces to that of[18] for the case where the modal "after-flow" constant reduces to zero. This treatment extends the time validity range to times of the order of shock wave transit time, while reducing accuracy for small values of time.

These investigations have been restricted to the two-dimensional problem, i.e., response to long shock waves. Longitudinal effects in the absence of cross sectional deformation are investigated in[33] and[34]. For this "whipping" mode, the model is a cylinder with rigid cross section, which is allowed to bend elastically about its longitudinal axis in response to a local (spherical) step wave. For the case of a symmetrical cylinder-wave situation (plane wave - cylinder),

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the results of Murray[34] compare in general with the translational ($n = 1$) mode of Mindlin - Bleich[18].

E. In-Vacuo Shell Normal Modes.

The natural frequencies for a circular cross-section in-vacuo, as applied in each of the previous approaches, are derived utilizing LaGrange's equations by Timoshenko[2]. The in-vacuo frequencies of vibration of a simply supported thin cylindrical shell, reinforced by equally-spaced, circular ring stiffeners, are derived by Galletly[43]. Bleich[45] has derived an approximate method for calculating the normal modes of ring-stiffened cylinders, found to agree with the more exact treatment of[43] within $\pm$ 10 per cent. These methods satisfactorily describe the natural frequencies of a more realistic model for lower modes without undue labor. The general form of the results of [45] are listed in Appendix D.

II. THEORETICAL PROCEDURE

A. Description of System Model.

The hull is characterized by a simple cylindrical shell with a circular cross section immersed in an infinite acoustic fluid. The response of the shell will be elastic. Mounted at Point A on the shell is a simple oscillator representative of a foundation - equipment chain, whose elastic response is to be determined. The equipment response is defined to be one-dimensional. Although the physical situation is defined to be two-dimensional (Fig. 1), longitudinal effects are compensated for in part by the use of an empirical factor, called the effective length ratio. No "whipping" effects are considered. The equipment - foundation response time, while remaining a variable in the solution quasi-impedance, is stated to be of the order of the shock wave transit time (5-6 msec.). The shell-simple oscillator combination is subjected to an acoustic, plane, exponentially-decaying step wave, normally incident to the longitudinal axis of the cylinder. The use of linearized theory is valid only for pressure waves which are not too intense and boundary excursions which are relatively small. Symmetric modal deformation of the cylindrical shell is assumed.

B. General Approach.

The sequence of events in the derivation of the cylinder quasi-impedance will be outlined before starting the mathematical details.

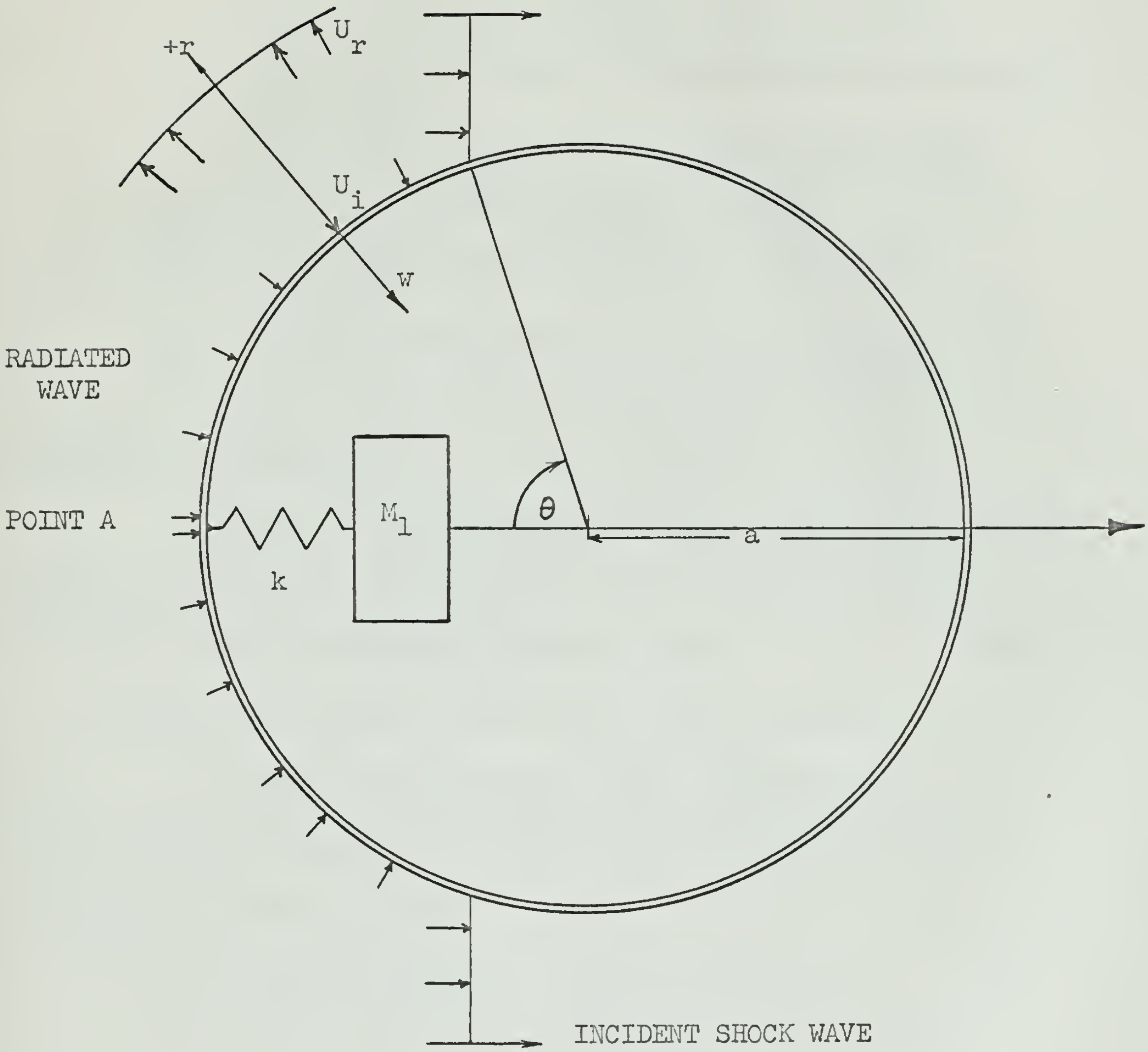

FIGURE 1. GEOMETRY

1. Define the shell response, the system input force and velocity, and a potential function for the radiated wave in terms of Fourier expansions of their generalized coordinates, generalized force and velocity, and modal potential function.
2. Define an effective modal mass of the shell.
3. Setup of the equation of motion of a point on the shell. The shell is immersed in the medium and the point feels the interaction with an equipment - foundation chain mounted inside. The shell is acted upon by the summation of the incident, radiated and diffracted pressure waves.
4. Description of the natural modal frequencies and shapes in-vacuo for the shell model chosen.
5. Approximate the characteristic impedance of the radiated wave.
6. Prescribe necessary boundary conditions.
7. Using (5) and (6), eliminate the fluid potential function from the modal equation of motion.
8. Express the modal equation of motion in terms of its Laplace transforms and convert to normal four-pole parameter format.
9. Set up the four-pole parameters for an appropriate model of the foundation - equipment chain.
10. Using matrix techniques, combine the shell and the equipment - foundation impedance functions.

11. Invert the resulting Laplacian expression of (10) to the time domain solving for the equipment response to one mode of the shell response to the input shock wave.
12. Sum the responses of the equipment to the modal excitations of the shell for as many modes as required.
13. Convert to response relative to hull if desired.

C. Impedance Parameters.

A general discussion of the mechanics of forming quadripole parameters for a wide variety of physical systems is available in[23]. The brief discussion included here is solely for the sake of continuity.

1. Definitions

A linear system which possesses a single input point, 1, and a single output point, 2, may be described by a pair of linear equations of the form

$$F_1 = e_{11} F_2 + e_{12} V_2 \qquad (1)$$

$$V_1 = e_{21} F_2 + e_{22} V_2$$

where e_{11}, e_{12}, e_{21}, and e_{22} are the quadripole or four-pole parameters of the system. Solving Equation (1) for the four poles gives:

$$e_{11} = \frac{F_1}{F_2} \text{ with } V_2 = 0\text{, i.e., station 2 blocked;}$$

$$e_{12} = \frac{F_1}{V_2} \text{ with } F_2 = 0\text{, i.e., station 2 free;}$$

$$e_{21} = \frac{V_1}{F_2} \text{ with } V_2 = 0\text{, i.e., station 2 blocked;}$$

$$e_{22} = \frac{V_1}{V_2} \text{ with } F_2 = 0\text{, i.e., station 2 free.}$$

The e_{11} parameter is thus the force transmissibility from station 1 to station 2 where station 2 is blocked. Parameter e_{12} is the transfer impedance between station 1 and station 2 with station 2 free. Parameter e_{21} is determined by the transfer impedance between stations 1 and 2 with station 2 blocked. Parameter e_{22} is defined in terms of the velocity transmittance from station 1 to station 2 with station 2 free. Equations (1) are conveniently expressed in matrix notation as:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} F_2 \\ V_2 \end{bmatrix} \qquad (2)$$

The first bracketed term on the right is called the quadripole parameter matrix. Matrix algebra provides a formalized and practical method of forming component quadripoles from fundamental elements, and of obtaining the overall system quadripole from its components.

2. Quadripole Impedances.

The elements of the four-pole matrix are naturally related to the various types of mechanical impedance that describe the component. The relationship between the two are listed below in order to provide correlation between the two methods of nomenclature.

The load or driving point impedance characterizes the elements that follow it:

$$e_{21} = \frac{V_1}{F_2} \text{ with } V_2 = 0, \text{ i.e., station 2 blocked.}$$

$$e_{22} = \frac{V_1}{V_2} \text{ with } F_2 = 0, \text{ i.e., station 2 free.}$$

The e_{11} parameter is thus the force transmissibility from station 1 to station 2 when station 2 is blocked. Parameter e_{12} is the transfer impedance between station 1 and station 2 with station 2 free. Parameter e_{21} is determined by the transfer impedance between stations 1 and 2 with station 2 blocked. Parameter e_{22} is defined in terms of the velocity transmission from station 1 to station 2 with station 2 free. Equations (1) are conveniently expressed in matrix notation as:

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The elements of the four-pole matrix are naturally related to the various types of mechanical impedance and, there- the components. The relationship between the two are listed below in order to provide correlation between the two systems of nomenclature.

The load or driving point impedance characteristics for elements that follow is:

$$Z_2 = F_2/V_2$$

The input or driving point impedance, defined by

$$Z_1 = \frac{F_1}{V_1}$$

is dependent on both the quadripole and on its load impedance. It is related to the load and the four-pole by

$$Z_1 = \frac{e_{11}\ Z_2 + e_{12}}{e_{21}\ Z_2 + e_{22}} .$$

The input transfer impedance is

$$Z_{12} = \frac{F_1}{V_2} = e_{11}\ Z_2 + e_{12} .$$

The force transfer function is

$$\frac{F_1}{F_2} = e_{11} + e_{12}/Z_2 .$$

The velocity transfer function is

$$V_1/V_2 = e_{22} + e_{21}\ Z_2$$

3. Forming the Parameters.

For some simple models direct combination of the known parameters for fundamental elements (Appendix C) using matrix algebra will produce the quadripole for the system. The definitions given above also provide a means of introducing experimental data into the analytic solution. However, in the general case it is necessary to:

(1) define the system

(2) describe the system in terms of a single pair of input and output junctions

(3) solve the performance equation using operational calculus methods

(4) resolve the equations into the form of equation (1) or (2).

D. Analysis of Shell-Shock Wave - Foundation - Equipment Interaction

1. Incident Wave.

The input is defined as an exponentially decaying step wave of the form

$$P_i = P_o\, e^{-\beta t}\, u(t).$$

The simple step is translated across the cylinder with the use of a time variant delay factor

$$\tau = \frac{a}{c}\,(1 - \cos\theta) \qquad \text{(origin taken at cylinder axis)}$$

so that

$$P_i = P_o\, e^{-\beta(t - \tau)}\, u\,(t - \tau) \tag{3}$$

The total pressure and the radial component of the particle velocity are expanded into Fourier series of the form

$$P_i = \sum_{n=0}^{\infty} P_{in} \cos n\,\theta$$

$$U_i = \sum_{n=0}^{\infty} U_{in} \cos n\,\theta \tag{4}$$

The Fourier modal coefficient P_{in} is obtained from equation (4) by modal orthogonality after multiplying right and left sides of equation (4) by cos n θ, integrating from 0 to 2π, interchanging the order of summation and integration. The integrand is zero from θ to 2π by the definition of the step, $u(t - \tau)$.

This procedure yields

$$P_{in} = \frac{1}{\pi} \int_0^{\theta(t)} P_i (\theta, t) \cos n \theta \qquad 0 < \theta (t) < 2\pi$$

where θ (t) is the angular position of the incident wave. Using the Laplace time - lag theorem for translation in the time domain, substituting from equation (4) and transforming with respect to the non-dimensional time T = ct/a yields,

$$\bar{P}_{in} = \frac{1}{\pi} \frac{e^{-s}}{(S + B)} \int_0^{\theta = \cos^{-1}(1-T)} e^{+ s \cos \theta} \cos n \theta \, d\theta \qquad 0 \leqslant \theta \leqslant 2\pi \tag{5}$$

Similar development applies to the radial modal particle velocity. Introducing an additional cos θ because of its radial definition and using the characteristic impedance of a plane wave,

$$U_{in} = \frac{1}{\pi} x \frac{1}{c} \int_0^{\theta(t)} \left[P_i (\theta, t) \cos n \theta \right] \cos \theta \, d\theta \qquad 0 \leqslant \theta \leqslant 2\pi$$

and

$$(s + B)\bar{U}_{in} = \frac{1}{\pi} x \frac{e^{-s}}{\rho c} \int_0^{\theta(t)} (e^{+ s \cos \theta} \cos n \theta) \cos \theta \, d\theta \qquad 0 \leqslant \theta \leqslant 2\pi \tag{6}$$

At $\theta = 2\pi$, the integral in equation (5) becomes a definition of the modified Bessel function. During transit time, they are referred to as incomplete modified Bessel functions. Their numerical evaluation is discussed in Appendix B. Using this definition, equations (5) and (6) reduce to:

$$\bar{P}_{in} = \frac{e^{-s}}{S + B} \epsilon_n I_{n\theta} (s) \tag{7}$$

$$\bar{U}_{in} = \frac{e^{-s}}{S + B} \epsilon_n \frac{P_o}{c} I'_{n\theta} (s)$$

(prime denotes derivative with respect to s.)

2. Interaction Pressure Field.

Contact between the incident wave and the cylinder produces scattered and radiated acoustic waves whose potential function must satisfy the wave equation. The scattered wave arises because of the presence of the cylinder. The radiated wave is caused by the response motion of the cylinder. The scattered and radiated waves are assumed to move radially out from the cylinder. The radiated pressure and velocity are defined in terms of the potential function Φ by

$$P_r = \rho \frac{\partial \Phi}{\partial t}$$

$$U_r = - \frac{\partial \Phi}{\partial r}$$

Modal contributions, corresponding to the in-vacuo generalized coordinates of cylinder normal modes, are obtained by expanding the velocity potential, thus P_r and U_r into a Fourier cosine series in θ of the form of equation (4). Equating coefficients of the expansions, yields:

$$P_{rn} = \rho \frac{\partial \Phi_n}{\partial t}$$

$$U_{rn} = - \frac{\partial \Phi_n}{\partial r}$$

where the generalized potential function also satisfies the wave equation in cylindrical coordinates

$$\frac{\partial^2 \phi_n}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_n}{\partial r} - \frac{n^2}{r^2}\phi_n = \frac{1}{c^2}\frac{\partial^2 \phi_n}{\partial t^2}$$

3. Cylindrical Wave Approximation.

The use of the complete form of the wave equation above to provide a relation between P_{rn} and U_{rn} (characteristic impedance) yields a solution whose inversion from the complex frequency domain is prohibitively tedious.

An approximation by Haywood[20] to the characteristic impedance of a cylindrical diverging wave yields results valid for times of the order of the shock wave transit time. Haywood's derivation yields:

$$\frac{\partial \phi_n}{\partial r} = -\frac{\partial \phi_n}{c\partial t} - \frac{\bar{g}_n}{r}\phi_n \tag{8}$$

or

$$\frac{\partial U_{rn}}{\partial t} = \frac{1}{\rho c}\frac{\partial P_{rn}}{\partial t} + \frac{\bar{g}_n}{\rho r}P_{rn}$$

The modal constant $\bar{g}_n$ is the average function whose limits, while time ranges between 0 and ∞, is:

$$0 < g_n < 1/2$$

$$1/2 < g_n < n \qquad \text{for } n > 1 \tag{9}$$

The main effect of the approximation is to give a final velocity, or "afterflow" to the fluid, following the passage of the pressure wave, which goes to zero only after the pressure has fallen below its equilibrium value. For distances infinitely far from the cylinder, equation (8) reduces to the plane wave approximation.

4. Natural Frequencies of Shell Model.

For survey purposes, the simple circular unstiffened shell model is considered adequate. Extensional effects in all modes except the dilatational ($n = 0$) are neglected. The derivation[2][15] of the normal mode frequencies for a simple circular cross section yields:

$$\omega_o^{\ 2} = \frac{\bar{E}}{\rho_s a^2} \qquad (n = 0) \qquad (10)$$

with $\bar{E} = \dfrac{E}{(1 - \nu^2)}$

$$\omega_n^{\ 2} \text{ (inextensional - flexural)} = \frac{n^2 (1 - n^2)}{1 + n^2} \times \frac{\bar{E}}{\rho_s} \times \frac{I}{A} \times \frac{1}{a^4}$$

The natural frequency of the translational mode $n = 1$ is, of course, equal to zero.

In order to reduce the order of the characteristic equation for a later approximation, we now define

$$\bar{\omega}_n = \frac{\omega_n}{m_n} \text{ as the modal shell stiffness,}$$

where $m_n = m$, ($n = 0$) and

$$m_n = \left(\frac{1 + n^2}{n^2}\right) m, \ (n > 0).$$

5. Equation of Motion at Point on Shell.

Summation of the modal forces acting at point A in Fig. 1 gives:

$$m_n \frac{\partial^2 q_n}{\partial t^2} + \bar{\omega}_n^{\ 2} q_n = P_{in} + P_{rn} + F'_{2n} \qquad (12)$$

where q_n is a generalized coordinate and is defined in terms of the total radial deflection w by:

$$w = \sum_{n=0}^{\infty} q_n \cos n\theta$$

6. Boundary Condition.

At any point on the surface of the shell, the modal velocity of the shell must be equal to the net particle velocity of the incident and radiated - diffracted modal waves

$$\dot{q}_n = U_{in} - U_{rn} \tag{13}$$

7. Net Shell Response Function.

An operational flow chart combining the preceding steps is given in Figure 2.

Transforming equations (8), (12) and (13) by Laplace transforms with respect to time yields

$$\left(m_n \frac{s^2 + \bar{\omega}_n^2}{s}\right) \bar{Q}_n = \bar{P}_{in} - \bar{F}'_{2n} + \frac{\rho c s}{s + \frac{g_n c}{a}} (\bar{U}_{in} - \bar{Q}_n) \tag{14}$$

Rearranging and changing the time variable to a non-dimensional time T (= ct/a) yields:

$$(\Delta) \bar{Q}_n = \frac{1}{\rho c} \left[-s(s + \bar{g}_n) \bar{F}'_{2n} + s\,(s + \bar{g}_n) \bar{P}_{in} + s^2 \rho c \,\bar{U}_{in} \right] \tag{15}$$

where Δ, the characteristic equation of the shell response, equals:

$$\frac{m_n}{\rho a} s^3 + \left(\frac{m_n}{\rho a} \bar{g}_n + 1\right)s^2 + \left(\frac{a}{\rho c^2} \bar{\omega}_n^2\right) s + \frac{a}{\rho c^2} \bar{\omega}_n^2 \bar{g}_n \tag{16}$$

where q_n is a generalized coordinate and is defined in terms of the total radial deflection w by:

$$w = \sum_{n=0}^{\infty} q_n \cos n\theta$$

6. Boundary Condition.

At any point on the surface of the shell, the modal velocity of the shell must be equal to the net particle velocity of the incident and radiated - diffracted modal waves

$$\dot{q}_n = U_{in} - U_{rn} \qquad (13)$$

7. Net Shell Response Function.

An operational flow chart combining the preceding steps is given in Figure 2.

Transforming equations (8), (12) and (13) by Laplace transforms with respect to time yields

$$\left(\frac{m_n s^2 + \bar{\Omega}_n^2}{s}\right) \bar{\dot{q}}_n = \bar{P}_{in} - \bar{P}_{rn} = \frac{\rho c s}{s + \frac{\beta_n c}{a}} \left(\bar{U}_{in} - \bar{\dot{q}}_n\right) \qquad (14)$$

Rearranging and changing the time variable to a non-dimensional time T (= ct/a) yields:

$$\Delta(s)\, \bar{\dot{q}}_n = \frac{1}{\rho c}\left[-s(s + \bar{\beta}_n)\, \bar{F}'_{in} + s\,(s + \bar{\beta}_n)\, \bar{F}_{in} + s^2 \rho c\, \bar{U}_{in}\right] \qquad (15)$$

where Δ, the characteristic equation of the shell response, equals:

$$\frac{m_n}{\rho a} s^3 + \left(\frac{m_n}{\rho a}\bar{\beta}_n + 1\right)s^2 + \left(\frac{a}{c^2}\bar{\Omega}_n^2\right)s + \frac{a}{c^2}\bar{\Omega}_n^2 \bar{\beta}_n \qquad (16)$$

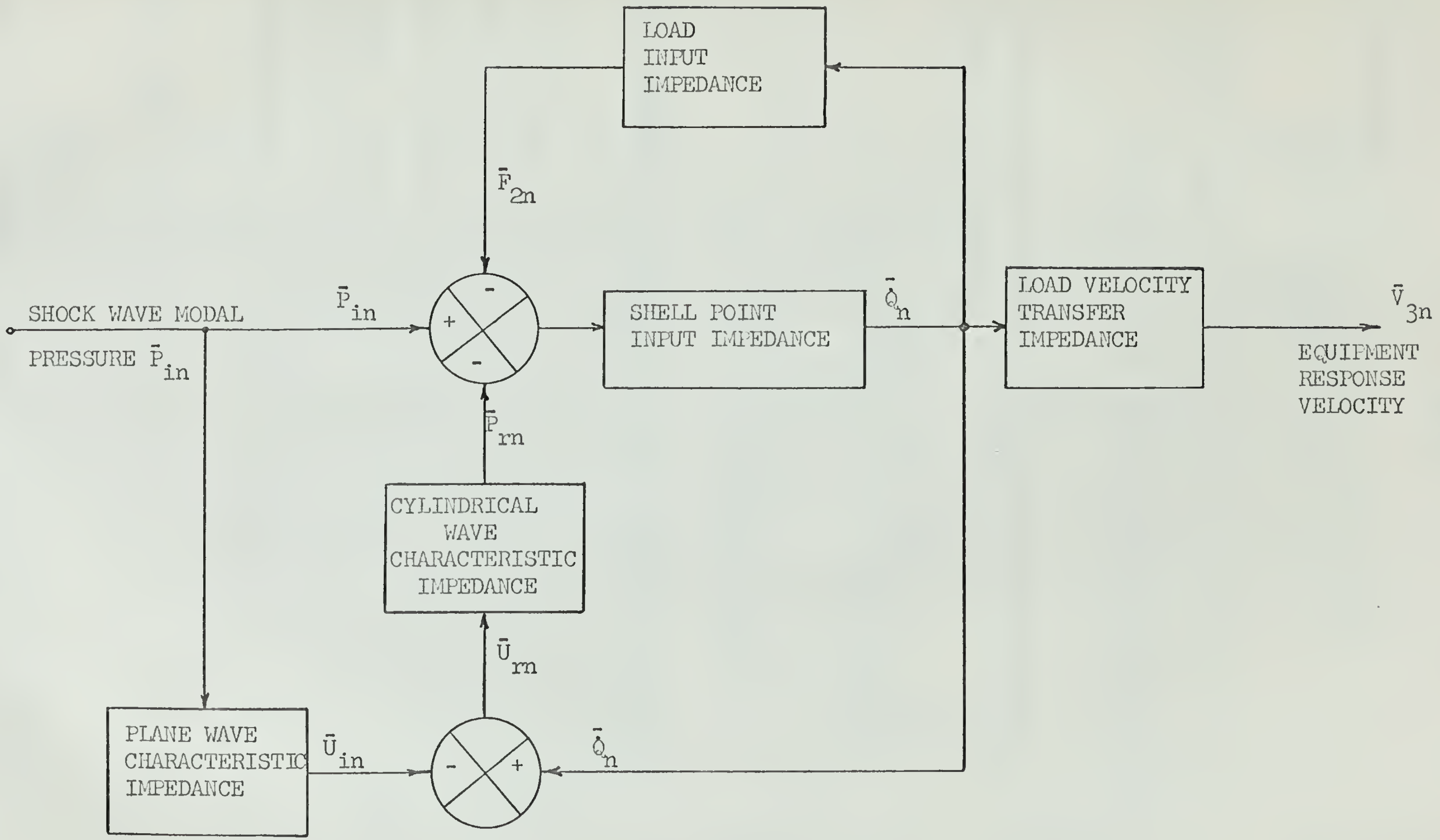

FIGURE 2

SHOCK WAVE - SHELL - EQUIPMENT INTERACTION FLOW CHART

From equation (7),

$$\frac{\bar{P}_{in}}{\bar{U}_{in}} = \rho c \, \frac{I_{n\theta}(s)}{I'_{n\theta}(s)}$$

therefore, equation (15) reduces to:

$$\rho c\,(\Delta)\,\bar{Q}_n = -s\,(s + \bar{g}_n)\,\bar{F}'_{2n} + \left[s(s + \bar{g}_n) + s^2 \frac{I'_n}{I_n}\right] \bar{P}_{in} \qquad (17)$$

Introducing the definition of $\bar{P}_{in}$ for step decaying shock wave equation (7), we obtain

$$P_o = \frac{s\,(s + B)(s + \bar{g}_n)}{e^{-s}\left[s\,(s + \bar{g}_n)\,I_{n\theta}(s) + s^2\,I'_{n\theta}(s)\right]}\,\bar{F}'_{2n}$$

$$+ \frac{\rho c\,\Delta\,(s + B)}{\left[s\,(s + \bar{g}_n)\,I_{n\theta}(s) + s^2\,I'_{n\theta}(s)\right]e^{-s}}\,\bar{Q}_n \qquad (18)$$

E. Equipment - Foundation and Shell Four-Pole Parameters.

In general, there is no limit to the complexity of the model of the equipment - foundation chain mounted at A in Figure 1, except that it be capable of representation by point attachment. For the purpose of this investigation, a simple oscillator was chosen. The mass and natural frequency may be varied to determine their effects on the shell - equipment interaction.

The mass four-pole as derived in Appendix C is:

$$\begin{bmatrix} 1 & sM_1 \\ 0 & 1 \end{bmatrix}$$

The spring four-pole is found to be:

$$\begin{bmatrix} 1 & 0 \\ s/k & 1 \end{bmatrix}$$

They are connected in cascade, thus are combined by matrix multiplication to give:

$$\begin{bmatrix} \bar{F}_{2n} \\ \bar{V}_n \end{bmatrix} = \begin{bmatrix} 1 & M_1 s \\ s/k & \dfrac{k + M_1 s^2}{k} \end{bmatrix} \begin{bmatrix} \bar{F}_{3n} \\ \bar{V}_{3n} \end{bmatrix} \tag{19}$$

This four-pole has been transformed with respect to time. In order to conform to equation (18), we change the time variable to non-dimensional form T $(= \frac{ct}{a})$. The four-pole becomes:

$$\begin{bmatrix} 1 & M_1 s \cdot \frac{c}{a} \\ \frac{c}{a} \cdot \frac{s}{k} & 1 + \dfrac{M_1 s^2 \cdot c^2/a^2}{k} \end{bmatrix} \tag{20}$$

We see that equation (18) fits the general form of one of the linear four-pole equations (1). There are several points to be discussed or rationalized before we give (18) an impedance tag.

(1) Equation (15) describes the modal response of a distributed parameter system to the spatially distributed, time variant force of the pressure loading and a dimensionally similar equipment reaction force. Implicit in the impedance concept is the necessity for point input - output description of the system. The application of the impedance concept can be rationalized if we consider as the input to point A an equivalent point force which would produce the same response at A as the distributed pressure loading. Attached to the output of point A is the equipment model.

(2) To further satisfy our point requirement for the cross section, we state that:

$$\begin{bmatrix} F_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & M s \\ \frac{\omega^2}{s} & \frac{k + M s^2}{k} \end{bmatrix} \begin{bmatrix} F_1 \\ v_1 \end{bmatrix} \qquad (19)$$

This four-pole has been transformed with respect to time. In order to conform to equation (15), we change the time variable to non-dimensional form T ($= \frac{dt}{\theta}$). The four-pole becomes:

$$\begin{bmatrix} 1 & R_1 s \cdot \frac{\theta}{s} \\ \frac{s}{\theta} \cdot \frac{s}{k} & 1 + \frac{R_1 s \cdot s^2/\theta^2}{k} \end{bmatrix} \qquad (20)$$

We see that equation (20) fits the general form of one of the linear four-pole equations (1). There are several points to be discussed or rationalized before we give (15) an impedance way.

(1) Equation (15) describes the modal response of a distributed parameter system to the spatially distributed, time variant force of the pressure loading and a dimensionally similar equivalent reaction force. Implicit in the impedance concept is the necessity for point input - output description of the system. The application of the impedance concept can be rationalized if we consider as the input to point A an equivalent point force which would produce the same response at A as the distributed pressure loading. Attached to the output of point A is the equivalent model.

(2) To further satisfy our point requirement for the cross section, we choose that:

$$\bar{F}_{2n} \text{ (equation (15)) } = \frac{F'_{2n}}{K_1} \qquad (21)$$

where $\bar{F}'_{2n}$ is an equivalent point force for the cross section, but remains a distributed force longitudinally representing the equipment reaction force per unit of longitudinal length.

$$K_1 = 2\pi a \, L_{eff} \qquad (22)$$

where L_{eff} is defined to be the ratio of the longitudinal length of shell feeling the influence of the equipment reaction, to the actual equipment length. L_{eff} is, of course, an empirical input. It will be a function of relative stiffnesses and masses, but a range of 2-5 is considered reasonable.

(3) The interaction function is defined here only for an exponentially decaying step wave. However, physically, this is quite acceptable, since by superposition and the use of the Laplace delay factor, the characteristics of any normal pressure loading can be simulated.

(4) The net impedance function is defined in terms of an infinite series of modal impedances.

(5) The complexity of the model prevents complete description of the four-pole parameters. The problem is in essence similar to the transmission line problem in electrical engineering. There, plane waves travelling along a transmission line, described by a characteristic impedance, meet an

interface or load impedance different from that of the transmission line. Due to the interaction, standing waves are set up in the transmission line altering the voltage - current relationship (input impedance). The analogy is made by regarding the transmission line as the medium surrounding the cylinder, and replacing voltage and current, by force and velocity. The essential difference and reason that the analogy can be extended no further arises from the fact that with the cylinder the problem is no longer bilateral. The input wave impedance differs from that of the output.

In order to form the four-pole parameters according to the standard recipe, we would first define the system components and component point inputs and outputs as:

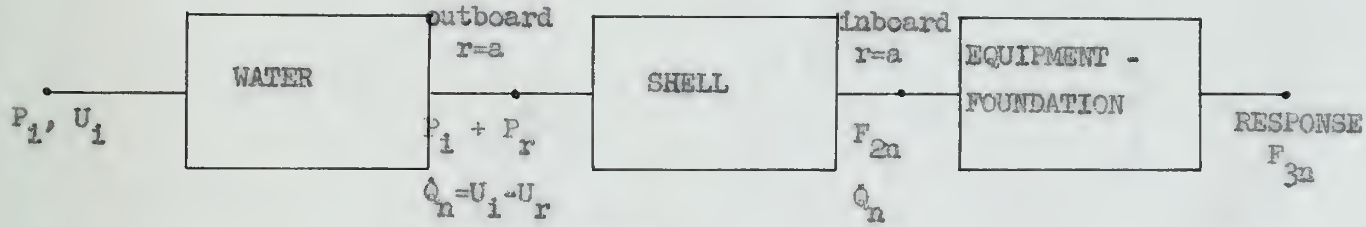

FIGURE 3

SYSTEM FLOW CHART

a. Plane wave characteristic impedance $\equiv \rho c$

$$\frac{P_i}{U_i} = \rho c \cos \theta$$

b) Input transfer impedance for the water component is:

$$\frac{P_i}{U_i - U_r}$$

c. Force transfer impedance for the water is:

$$\frac{P_i}{P_i + P_r} \; .$$

d. Velocity transfer impedance for the water is:

$$\frac{U_i}{U_i - U_r} \; .$$

e. Load impedance at the water output junction is:

$$\frac{F_{2n}}{Q_n}$$

It is possible to describe

$$\frac{P_i}{P_i + P_r} \text{ and } \frac{P_i}{U_i - U_r}$$

or

$$\frac{U_i}{P_i + P_r} \quad \text{and} \quad \frac{U_i}{U_i - U_r}$$

using equations (18), (7), (13), but we have not adequately described the interaction to be able to define both without obtaining one redundant set. However, for cases where the response of the last element in a chain is desired, i.e., output end is free, force equal to zero, one of the two four-pole equations is adequate to describe the motion response. All interactions enroute are still accounted for. Such is the case in this example.

Thus, we have not described the system components entirely. In fact, we have shunted the complicated interaction at the shell surface and have dealt with a system whose limits are defined at the input by the free stream shock wave and at the output by the

inboard junction to the equipment - foundation four-pole.

With these rather restrictive conditions governing our transfer function, we call it a quasi-impedance.

The shell function equation (18) is now combined with the four-pole of (20) and definition equation (21) to produce after some rearrangement:

$$\bar{V}_{3n}(s) = \frac{s\, \epsilon_n\, P_o \left[(s + \bar{g}_n)\, I_{n\theta}(s) + s\, I'_{n\theta}(s)\right] e^{-s}}{(s + B)\left[s^2 \frac{c}{a}(s + \bar{g}_n)\frac{M_1}{K_1} + \frac{\rho c}{k}\left(M_1 s^2 \frac{c^2}{a^2} + k\right)\Delta\right]} \tag{23}$$

where $$\Delta = \frac{m_n}{\rho a} s^3 + \left(\frac{m_n}{\rho a}\bar{g}_n + 1\right) s^2 + \left(\frac{a}{\rho c^2}\bar{\omega}_n^{\,2}\right) s + \frac{a}{\rho c^2}\bar{\omega}_n^{\,2}\, g_n \tag{24}$$

Finally, after defining

$$\omega_{mach}^2 = \frac{k}{M_1} = \text{natural frequency of foundation - equipment} \tag{25}$$

$$V_o = \frac{P_o}{\frac{c^3}{a^3} \times \frac{M_1}{k} \times m_n} = \frac{P_o}{\frac{c^3}{a^3} \times \frac{1}{\omega_{mach}^2} \times m_n} \tag{26}$$

and expanding

$$\bar{V}_{3n}(s) = \frac{s\, \epsilon_n\, V_o \left[(s + \bar{g}_n)\, I_{n\theta}(s) + s\, I'_{n\theta}(s)\right] e^{-s}}{(s + B)(s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o)} \tag{27}$$

with $$a_4 = \left(\bar{g}_n + \frac{\rho a}{m_n}\right)$$

$$a_3 = \frac{a^2}{c^2}\left(\frac{\bar{\omega}_n^{\,2}}{m_n} + \omega_{mach}^2 + \frac{1}{K_1}\,\frac{k}{m_n}\right)$$

$$a_2 = \frac{a^2}{c^2} \left(\frac{\bar{\omega}_n^2}{m_n} \bar{g}_n + \omega_{mach}^2 \bar{g}_n + \rho a \frac{\omega_{mach}^2}{m_n} + \frac{1}{K_1} \; \frac{k}{m_n} \; \bar{g}_n \right)$$

$$a_1 = \frac{a^4}{c^4} \; \frac{\bar{\omega}_n^2}{m_n} \; \omega_{mach}^2$$

$$a_0 = \frac{a^4}{c^4} \; \frac{\bar{\omega}_n^2}{m_n} \, \omega_{mach}^2 \, \bar{g}_n$$

and the roots of the fifth order characteristic equation equal to s_i, $i = 1, 5$.

After eliminating the Bessel function derivative in (27) using the following recurrence relation:

$$I_n' \, (\alpha s) = \frac{d}{ds} \, I_n \, (\alpha s) = \alpha \, I_{n-1} \, (\alpha s) - \frac{n}{s} \, I_n \, (\alpha s) \qquad (28)$$

the velocity is inverted from the complex frequency domain. Use of the residue theorem from complex variable theory (4) makes the procedure quite straightforward.

The velocity response for all modes other than translational ($n = 1$) takes the form:

$$V_3 \, (T) = \frac{\epsilon_n \, V_o \, (-B) \left[(\bar{g}_n - n - B) \, I_{n\theta} \, (-B) - (B) \, I_{(n-1)_\theta}(-B) \right] e^{B(1-T)}}{(s_1 + B)(s_2 + B)(s_3 + B)(s_4 + B)(s_5 + B)}$$

$$+ \, \epsilon_n \, V_o \sum_{i=1}^{5} \frac{(s_i) \left[(g_n - n + s_i) I_{n\theta}(s_i) + s_i \, I_{(n-1)_\theta}(s_i) \right]}{H_i} \, e^{-s_i(1 - T)} \qquad (29)$$

where $H_1 = (s_1 + B)(s_1 - s_2)(s_1 - s_3)(s_1 - s_4)(s_1 - s_5)$

$$H_2 = (s_2 + B)(s_2 - s_1)(s_2 - s_3)(s_2 - s_4)(s_2 - s_5)$$

$$H_3 = (s_3 + B)(s_3 - s_1)(s_3 - s_2)(s_3 - s_4)(s_3 - s_5)$$

$$H_4 = (s_4 + B)(s_4 - s_1)(s_4 - s_2)(s_4 - s_3)(s_4 - s_5)$$

$$H_5 = (s_5 + B)(s_5 - s_1)(s_5 - s_2)(s_5 - s_3)(s_5 - s_4)$$

In the translational mode ($n = 1$), the shell stiffness, $\bar{\omega}_n^2$, goes to zero. The coefficients of the characteristic equation,

$$s^2\,(s^3 + a_4\, s^2 + a_3\, s + a_2),$$

reduce to

$$a_4 = \bar{g}_n + \frac{\rho a}{m_n}$$

$$a_3 = \frac{a^2}{c^2}\left(\omega_{mach}^2 + \frac{1}{K_1}\,\frac{k}{m_n}\right) \tag{30}$$

$$a_2 = \frac{a^2}{c^2}\left(\omega_{mach}^2\,\bar{g}_n + \rho a\,\frac{\omega_{mach}^2}{m_n} + \frac{1}{K_1}\,\frac{k}{m_n}\,\bar{g}_n\right)$$

$$a_1 = 0$$

$$a_0 = 0$$

The calculation of the residues must be repeated due to the double pole at the origin. The translational velocity of the oscillator mass is found to be of the form:

$$\begin{aligned} V_{3n} = {} & + 2\,V_o\left[\frac{(g_1 - 1)\,I_{1\theta}\,(0)}{a_2\,B}\right](1 + T) \\ & + 2\,V_o\left[\frac{(1 + B - \bar{g}_n)\,I_{1\theta}\,(-B) - B^2\,I_{0\theta}\,(-B)}{B^2\,(s_1 + B)(s_2 + B)(s_3 + B)}\right]e^{B(1 - T)} \\ & + \sum_{i=1}^{3} 2\,V_o\left[\frac{(s_i + \bar{g}_n - 1)s_i\,I_{1\theta}\,(s_i) + s_i^{\,2}\,I_{0\theta}\,(s_i)}{(s_i)^2\,(s_i + B)\,H_i}\right]e^{-s_i(1 - T)} \end{aligned} \tag{31}$$

where $H_1 = (s_1 - s_2)(s_1 - s_3)$

$$H_2 = (s_2 - s_1)(s_2 - s_3)$$

$$H_3 = (s_1 - s_3)(s_2 - s_3)$$

The velocity of the equipment mass as derived here is in an absolute reference space. The absolute velocity is a satisfactory measure of the effects of system and component parameter variation. However, from the standpoint of equipment damage criterion, a more useful quantity is the relative motion between the hull and the equipment. Relative motion will, of course, govern the maximum stress, deflection, and acceleration that the equipment and foundation will see. The power of the impedance technique is revealed in the simplicity with which the relative velocity is calculated.

(1) Equation (27) gives the equipment modal velocity in frequency (s) domain,

$$\bar{V}_{3n}(s) = \frac{P_o}{G(s)}$$

(2) Equipment - foundation four-pole parameters given as:

$$\begin{bmatrix} \bar{F}_{2n} \\ \bar{V}_{2n} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \bar{F}_{3n} \\ \bar{V}_{3n} \end{bmatrix} \qquad (2)$$

(3) Since $\bar{F}_{3n} = 0$,

$$\bar{V}_{2n}(s) = e_{22}(s)\ \bar{V}_{3n}(s) \qquad (32)$$

(4) Therefore,

$$\bar{V}_{2n}(s) = e_{22}(s) \times \frac{P_o}{G(s)}$$

(5) Inversion to time domain, if desired, is identical in procedure to that of $\bar{V}_{3n}(s)$. The poles of the

response function are of course identical since the characteristic equation for the system is unchanged.

(6) The relative velocity is then:

$$\bar{\dot{X}}\ (s) = \bar{V}_{3n}\ (s) - \bar{V}_{2n}\ (s) \quad \text{or,}$$

$$\dot{x}\ (T) = V_{3n}\ (T) - V_{2n}\ (T). \qquad (33)$$

F. Numerical Procedure.

Equations (29) and (31) will form the basis of the time response calculations. While the entire procedure is straightforward, the number of cycles required to trace the time history, coupled with the superposition of an indeterminant number of shell modes, and the large amounts of complex number arithmetic, lead to the use of a digital computer. Since the general form of the residues is determined, the machine needs only to:

(1) accurately determine the roots, real and complex, of the characteristic equation.

(2) perform numerical quadrature or series approximation (Appendix B) to the incomplete Bessel functions at steps sufficiently small to provide time history desired

(3) evaluate equations (29) and (31) as functions of time

(4) superpose modal responses.

The frequency response ($s = i\omega$) of the quasi-impedance is readily available by choosing the range and discrimination necessary.

response function are of course identical since the characteristic equation for the system is unchanged.

(5) The relative velocity is then:

$$\dot{\lambda}(s) = \bar{v}_{m}(s) - \bar{v}_{pm}(s) \quad \text{or,}$$

$$\dot{\lambda}(T) = V_{m}(T) - V_{pm}(T). \tag{31}$$

V. Numerical Procedure.

Equations (29) and (31) will form the basis of the time response calculations. While the entire procedure is straightforward, the number of cycles required to trace the time history, coupled with the superposition of an indeterminant number of shell modes, and the large amounts of complex number arithmetic, lead to the use of a digital computer. Once the general form of the response is determined, the computer needs only to:

(1) accurately determine the roots, real and complex, of the characteristic equation.

(2) perform numerical quadrature or series approximation (Appendix A) to the incomplete Bessel functions at steps sufficiently small to provide time history control.

(3) evaluate equations (29) and (31) as functions of time

(4) superpose modal responses.

The frequency response (s = iω) of the shell-impedance is readily available by choosing the roots and [illegible] necessary)

Once satisfactorily programmed, all parameters can, of course, be varied, using systematic procedures to determine existing trends. For the initial computer runs, typical parameters were arbitrarily chosen to characterize the shell model, and shock wave. Three separate combinations of equipment - foundation model mass and natural frequency were chosen in a typical range to order to evaluate the ability of the quasi-impedance to describe interaction effects.

So as to be able to check by hand initial calculations using a closed form root-finder, the order of the characteristic equation was reduced from quintic to quartic. This reduction was accomplished by neglecting the shell mass. The maximum error will be experienced for times shortly after the shock wave arrival. The natural periods of the equipment models chosen, are of the order of transit time. However, it is obvious from equation (28) that the complete interaction is a function of relative hull - equipment masses. Therefore, no physical justification is offered, but rather the argument that the general ability or inability of the function to characterize the interaction will be retained. Details of the programming procedures are listed in Appendix A. The computer language is IBM FORTRAN and the computer IBM 7090.

III. DISCUSSION OF RESULTS

A quasi-impedance function has been derived for a simple, but physically reasonable model. The function possesses flexibility for extension to analytic models of a more complex nature without changes in its basic form. It further provides an opportunity for incorporation of empirical data in the form of measured component impedances and shell geometry variables into an analytic description on a rational basis. The function describes the equipment model response in terms of physically meaningful parameters:

a. equipment - foundation mass, stiffness, and natural frequency or experimentally determined impedances

b. shell mass

c. shell in-vacuo natural frequencies

d. shell geometry

   1. radius

   2. cross-section moment of inertia

   3. shell thickness

   4. frame spacing, area and moment of inertia with the use of Appendix D

   5. empirical input of "effective length ratio"

e. shock wave characteristics

   1. peak pressure

   2. exponential decay constant.

Complete evaluation can, of course, be made only after completion of a program of systematic parameter variation in

III. DISCUSSION OF RESULTS

A shock-impedance function has been derived for a simple, but physically reasonable model. The function possesses flexibility for extension to analytic models of a more complex nature without changes in its basic form. It further provides an opportunity for incorporation of empirical data in the form of measured component impedances and shell geometry variables into an analytic description on a rational basis. The function describes the equipment modal response in terms of physically meaningful parameters.

a. equipment - foundation mass, stiffness, and natural frequency or experimentally determined impedances

b. shell mass

c. shell in-vacuo natural frequencies

d. shell geometry

1. radius
2. cross-section moment of inertia
3. shell thickness
4. frame spacing, mass and moment of inertia with the use of Appendix D
5. empirical input of "effective length ratio"

e. shock wave characteristics

1. peak pressure
2. exponential decay constant.

Complete evaluation can, of course, be made only after completion of a program of systematic parameter variation as

conjunction with comparison with measured field data from realistic tests. The numerical procedure to test the general validity of this approach during this exploratory phase is outlined in II-F and in Appendix A. Conclusion of the time allotted for the numerical example occurred before the digital computer program was completely functional. Complete reliability of the root-finder subprogram to obtain the real and complex roots of the characteristic equation was not achieved. Two methods using the digital program and another hand-calculated test method were used in searching for the roots. Comparison of the results from these three methods lead the author to conclude that a combination of absolute and relative root magnitudes in the lower modes and relative root magnitudes in the higher modes both prevent convergence and introduce numerical inaccuracies into the root-finding technique. While the possibility of programmer procedural error is not precluded, careful checking has revealed no blunders (as opposed to errors).

The major program using a double precision arithmetic method for automatic computation in a complex number mode of the pole residues and time variance appeared to function properly. The failure to achieve general program reliability naturally prevents inclusion of the velocity responses calculated. However, for the modes whose characteristic equation roots were reasonably accurate, responses were of a damped oscillatory form as would be physically expected. As the weight and natural frequency of the equipment were varied, response calculated conformed as expected (25); i.e., the larger the mass, the lower the peak mass velocity

and the larger the rise time. The influence of mass of equipment on peak velocity can be seen directly in equations (27) and (29). The direction of the mass effect on the rise time is less obvious, but it is clear that the mass influence is directly proportional to its magnitude. The accuracy of evaluation of the incomplete Bessel function for large negative root values during early times of the transit period became progressively worse as the mode order increased. The cause of this is shared by the massless shell used to simplify root-finding problems and the form of the equation for small times involving small differences between large numbers. Accurate results for these early phases of the response motion can be achieved by series expansion of the impedance in s and subsequent term by term integration. The quasi-impedance would then be described by a two-phase function, each phase with a time validity range. The need for this added complexity needs to be evaluated.

Equation (27) describes the velocity response relative to a set of fixed coordinates, which is adequate for some parametric studies. However, in most situations the damage evaluation or "performance level" is better described in terms of response motions relative to the hull. The utility of the impedance approach is demonstrated by the simplicity with which the relative velocity is obtained, as shown by equations (32) and (33).

In general, the investigator is convinced that the system model is ultimately capable of providing a qualitative and quantitative estimate of the response of a real cross section over a range of test conditions. It provides an analytic model bearing some of the

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Equation (47) describes the velocity response relative to a set of fixed coordinates, which is adequate for many parametric studies. However, in most situations the design evaluation or "performance level" is better described in terms of response motions relative to the hull. The utility of the impedance approach is demonstrated by the simplicity with which the relative velocity is obtained, as shown by equations (32) and (33).

In general, the investigator is convinced that the simple model is ultimately capable of providing a meaningful and quantitative estimate of the response of a real shock machine over a range of test conditions. It provides an analytic model bearing some of the

characteristics of a distributed mass system with the convenience of a discrete impedance chain system. As such it provides a means of attaching quantitative meaning, in the form of time response, to:

(1) shock-spectrum null seen at equipment fixed-base natural frequency in discrete mass experiments investigating distributed mass phenomena.

(2) knowledge that shock spectrum - normal mode approach, by destroying time history of input motion, results in over-conservative estimated response (due to neglecting of modal time to peak in summation process).

By describing the response in terms of physically meaningful parameters, the author believes that this approach will allow a better understanding of the basic phenomena. In providing a method of breaking down the complicated system interaction into component interactions capable of description by engineering approximation, the impedance approach is of great value.

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By describing the response in terms of physically meaningful parameters, the author believes that this approach will allow a better understanding of the basic phenomena. In providing a method of breaking down the complicated system interaction into component interactions capable of description by understandable approximation, the impedance approach is of great value.

CONCLUSIONS

1. It is possible to formulate an analytic approach describing the water applied pressure loading - hull - equipment interaction with the use of engineering assumptions, capable of solution with a reasonable amount of effort.
2. The use of mechanical impedance and transform methods provides a powerful analytic tool and a flexible format and procedure for the introduction of more exact model components and of empirical data.

RECOMMENDATIONS

1. Systematic parameter variation of the derived quasi-impedance function should be conducted to provide rational basis for the establishment of design inputs and goals.
2. The impedance function should be expanded to include the effects of:
   a. multi-degree of freedom equipment - foundation model
   b. foundation and equipment damping
   c. longitudinal whipping response
   d. experimental - empirical inputs
3. Response variation should be investigated as the equipment attachment point varies around the periphery.
4. Investigate need for two-phase impedance function to extend accuracy during early times of transit.
5. In conjunction with 2c and 3, investigate the effect of attack angle of incidence.
6. A quantitative investigation of the "spectrum dip" phenomenon for a distributed mass model of the form developed here should be conducted.

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5. In conjunction with 2c and 3, investigate the effect of oblique angle of incidence.
6. A quantitative investigation of the 'spectrum dip' phenomenon for a distributed mass model of the type developed here should be conducted.

APPENDIX

APPENDIX A.

DETAILS OF PROCEDURE

1. Computer Procedure.

   a. In order to expedite hand calculation checking of the IBM 7090 computer program during the exploratory phase of the problem, the order of the characteristic equation Δ is reduced by neglecting the shell mass. As a result, Δ reduces to:

$$s^2 + \frac{a}{\rho c^2}\,\bar{\omega}_n^2\, s + \frac{a}{\rho c^2}\,\bar{\omega}_n^2$$

or

$$s^2 + \sigma_n\, s + \sigma_n\, \bar{g}_n \quad \text{where}$$

σ_n is the non-dimensional shell stiffness

$$= \bar{\omega}_n^2\, \frac{a}{\rho c^2}$$

Equation (27) becomes:

$$\bar{V}_{3n} = \frac{s\, \sigma_n\, P_o \left[(s + \bar{g}_n) I_{n\theta}(s) + s\, I'_{n\theta}(s)\right] e^{-s}}{(s+B)(a_4\, s^4 + a_3\, s^3 + a_2\, s^2 + a_1\, s + a_o)}$$

where $\quad a_4 = \rho c \cdot \dfrac{M_1}{k} \cdot \dfrac{c^2}{a^2}$

$$a_3 = \rho c \cdot \frac{M_1}{k} \cdot \frac{c^2}{a^2} \cdot \sigma_n + M_1 \cdot \frac{c}{a} \cdot \frac{1}{K_1}$$

$$a_2 = \rho c \cdot \frac{M_1}{k} \cdot \frac{c^2}{a^2} \cdot \sigma_n \cdot \bar{g}_n + \rho c$$

$$+ M_1 \cdot \frac{c}{a} \cdot \bar{g}_n \cdot \frac{1}{K_1}$$

$$a_1 = \rho c\ \sigma_n$$

$$a_o = \rho c\ \sigma_n\ \bar{g}_n\ .$$

Rearranging, with

$$\omega^2_{mach} = \frac{k}{M_1} \quad \text{natural frequency of foundation equipment}$$

$$K_2 = K_1 \times \rho c^2$$

$$V_o = P_o \cdot \frac{a^2\ \omega^2_{mach}}{\rho c^3}$$

yields

$$\bar{V}_{3n} = \frac{V_o\ s\ c_n \left[(s + \bar{g}_n) I_{n\theta}(s) + s\ I'_{n\theta}(s)\right] e^{-s}}{(s+B)\left[s^4 + b_3\ s^3 + b_2\ s^2 + b_1\ s + b_o\right]} \qquad \text{(A-1)}$$

where $b_3 = \left[\sigma_n + (\frac{M_1}{K_2})(\omega^2_{mach})\right]$

$$b_2 = \left[\sigma_n\ \bar{g}_n + \frac{M_1}{K_2}\ \omega^2_{mach}\ \bar{g}_n + \frac{a^2}{c^2}\ \omega^2_{mach}\right]$$

$$b_1 = \frac{a^2}{c^2}\ \omega^2_{mach}\ \sigma_n$$

$$b_o = \frac{a^2}{c^2}\ \omega^2_{mach}\ \sigma_n\ \bar{g}_n\ .$$

The roots to the characteristic equation above will be called s_1, s_2, s_3, s_4.

Inversion of (A-1) is accomplished by calculating the residues about each of the poles, s_i.

The general form of the time response for all modes other than translational, is:

$$V_{3n}(T) = V_o\, \epsilon_n(-B) \left[\frac{(\bar{g}_n - n - B) I_{n\theta}(-B) - B\,(I_{(n-1)\theta}(-B))}{(s_1 + B)(s_2 + B)(s_3 + B)(s_4 + B)}\right] e^{B(1-T)}$$

$$+ V_o\, \epsilon_n(s_1) \left[\frac{(\bar{g}_n - n + s_1) I_{n\theta}(s_1) + s_1\, I_{(n-1)\theta}(s_1)}{(s_1 + B)(s_1 - s_2)(s_1 - s_3)(s_1 - s_4)}\right] e^{-s_1(1-T)}$$

$$+ V_o\, \epsilon_n(s_2) \left[\frac{(\bar{g}_n - n + s_2) I_{n\theta}(s_2) + s_2\, I_{(n-1)\theta}(s_2)}{(s_2 + B)(s_2 - s_1)(s_2 - s_3)(s_2 - s_4)}\right] e^{-s_2(1-T)}$$

$$+ V_o\, \epsilon_n(s_3) \left[\frac{(\bar{g}_n - n + s_3) I_{n\theta}(s_3) + s_3\, I_{(n-1)\theta}(s_3)}{(s_3 + B)(s_3 - s_1)(s_3 - s_2)(s_3 - s_4)}\right] e^{-s_3(1-T)}$$

$$+ V_o\, \epsilon_n(s_4) \left[\frac{(\bar{g}_n - n + s_4) I_{n\theta}(s_4) + s_4\, I_{(n-1)\theta}(s_4)}{(s_4 + B)(s_4 - s_1)(s_4 - s_2)(s_4 - s_3)}\right] e^{-s_4(1-T)}$$

$(n \neq 1.)$ (A-2)

b. Shell and equipment - foundation parameters were introduced as data to the machine, which calculated the modal coefficients of the characteristic equation of (A-1) for the first twenty modes. Subroutine root-finder was programmed to search for the real - complex roots of the resulting quartic. The first sub-program used was an IBM SHARE Distribution Program #1124, "MULLER"*. Persisting machine overflows, due perhaps to coefficient incompatibility, resulted in the author programming the Euler - Descartes Method

*"MULLER" is based upon the theory of Muller, J., "A Method for the Solution of Equations with Automatic Computers", Mathematical Tables of Automatic Computation, 1956, pp 208, 215.

for closed form solution of quartic, previously used to check computer results. The basic steps are:

1. reduce quartic

   $$y^4 + py^3 + qy^2 + ry + s = 0$$

   to form

   $$x^4 + ax^2 + bx + c = 0$$

   by substitution

   $$y = (x - p/4).$$

   We find

   $$a = \frac{8q - 3p^2}{8}$$

   $$b = \frac{16r - 8pq + 2p^3}{16}$$

   $$c = \frac{-3p^4 + 16qp^2 - 64rp + 256\ s}{256}$$

2. With ℓ, m, and n as the roots of the resolvent cubic:

   $$t^3 + (\frac{a}{2})\ t^2 + (\frac{a^2 - 4c}{16})\ t - \frac{b^2}{64} = 0$$

   The reduced quartic roots are:

   $$x_1 = \pm (- \sqrt{\ell} - \sqrt{m} - \sqrt{n})$$

   $$x_2 = \pm (- \sqrt{\ell} + \sqrt{m} + \sqrt{n})$$

   $$x_3 = \pm (\sqrt{\ell} - \sqrt{m} + \sqrt{n})$$

   $$x_4 = \pm (\sqrt{\ell} - \sqrt{m} - \sqrt{n})$$

   where upper sign is used if b is positive and the lower sign if b is negative.

c. Having calculated the roots of the characteristic equation, the computer was programmed to evaluate the modal incomplete Bessel function $I_{n\theta}$ of equation (B-1) at three-degree increments in envelopment angle with a three-station Simpson's Rule. After storing these values, the residues of equation (A-2) were evaluated at intervals of the non-dimensional time T corresponding to the envelopment angle (θ) increments of the integrated Bessel function.

2. Typical Shell and Equipment - Foundation Parameters Used.

a. Using typical values for E, I, a, ρ, ρ_s, m_n, yielded for the simple shell,

$$\omega_o^2 \text{ (extensional)} = (1.13 \times 10^6) \frac{\text{rad}^2}{\text{sec}^2},$$

$$\omega_n^2 \text{ (flexural)} = \frac{n^2 (1 - n^2)^2}{1 + n^2} \times 782 \frac{\text{rad}^2}{\text{sec}^2},$$

$$\bar{\omega}_o^2 = 4.23 \times 10^6 \ \ lb_f/ft^2 \text{ surface/ft length}$$

$$\bar{\omega}_n^2 = 3035.\ (1 - n^2)^2 \ \ lb_f/ft^2 \text{ surface/ft length.}$$

b. Three combinations were chosen for the equipment - foundation parameters:

1. M_1 = 155.2 slugs/ft.
   k = 5.515×10^6 lb_f/ft.
   f = 30 cycles per second (cps)

2. M_1 = 7.76 slugs/ft.
   k = 1.960×10^6 lb_f/ft.
   f = 80 cps

3. M_1 = 155.2 slugs/ft.

   k = 88.2×10^6 lb_f/ft.

   f = 120 cps

c. Shock Wave Characteristics.

   A peak pressure of 2000 psi and a decay constant $\beta = 20.8 \text{ sec}^{-1}$ were introduced as data to the machine.

APPENDIX B.

NUMERICAL EVALUATION OF INCOMPLETE AND COMPLETE MODIFIED BESSEL FUNCTION OF THE FIRST KIND

During the shock wave transit across the cylinder, evaluation of the Fourier modal coefficients of the incident pressure yields the function:

$$I_{n\theta}(s) = \frac{1}{\pi} \int_0^\theta e^{s \cos \theta} \cos n\theta \, d\theta \qquad 0 \leqslant \theta \leqslant 2\pi \qquad \text{(B-1)}$$

where θ is the angle of envelopment. For times greater than the transit time of the shock wave, $I_{n\theta}$ becomes:

$$I_n(s) = \frac{1}{\pi} \int_0^{2\pi} e^{s \cos \theta} \cos n\theta \, d\theta \, . \qquad \text{(B-2)}$$

Equation (C-2) is the definition of the modified Bessel function of order n. Equation (C-1) is therefore termed the incomplete modified Bessel function of order n. Equation (C-1) is not integrable in closed form, but is capable of evaluation by:

(1) numerical quadrature

(2) rapidly convergent series approximation.

Figure B-1 plots $I_{n\theta}$ (s) as function of T (or equivalently θ) for a specified real root of the characteristic equation, as obtained using three-station Simpson's Rule. It is presented to give an idea of the gross form of the function. In general, s is complex and requires integration of real and imaginary parts. Equation (B-1) takes the form:

$$I_{n\theta}(s) = \frac{1}{\pi}\int_0^{\theta} e^{(x+iy)\cos\theta}\cos n\theta\, d\theta$$, which can be

written by Euler's identity as:

$$\frac{1}{\pi}\int_0^{\theta} e^{x\cos\theta}\left[\cos(y\cos\theta) + i\sin(y\cos\theta)\right]\cos n\theta\, d\theta ,$$

or

$$\frac{1}{\pi}\int_0^{\theta} e^{x\cos\theta}\cos(y\cos\theta)\cos n\theta\, d\theta$$

$$+ i\frac{1}{\pi}\int_0^{\theta} e^{x\cos\theta}\sin(y\cos\theta)\cos n\theta\, d\theta \qquad \text{(B-3)}$$

where x and y refer to the real and imaginary parts of s respectively. A digital machine makes the evaluation of (B-3) quite effortless.

An alternate method of evaluation is to use the Jacobi-Anger formula and a series expansion of the modified Bessel function. Bessel functions of non-negative order n = 0, 1, 2.... are single valued integral functions of s. They may be formed as coefficients of the Fourier series

$$e^{+z\cos t} = \sum_{n=0}^{n=\infty} \epsilon_m I_n(s)\cos n t \qquad \text{(Jacobi-Anger Formula)} \qquad \text{(B-4)}$$

where ϵ_n (Neumann factor) = 1 for n = 0

= 2 for n = 1, 2,... .

Using (B-4), (B-1) goes to:

$$\pi I_{n\theta}(s) = \int_0^{\theta}\cos n\theta \sum_{m=0}^{\infty}\epsilon_m I_m(s)\cos m\theta\, d\theta .$$

The $\epsilon_m\ I_m\ (s)$ is not a function of the integration, so that we can bring it outside the summation sign and interchange the integration and summation signs yielding a form:

$$\pi\ I_{n\theta}(s)\ I_0'(s) \int_0^\theta \cos n\theta\ d\theta + 2\ I_1\ (s) \int_0^\theta \cos n\theta \cos\theta\ d\theta$$

$$+ 2I_2\ (s) \int_0^\theta \cos n\theta \cos 2\theta\ d\theta$$

$$+ 2I_3\ (s) \int_0^\theta \cos n\theta \cos 3\theta\ d\theta$$

$$+ \ldots\ldots \quad . \tag{B-5}$$

Equation (B-5) involves for real s only tabulated functions and integrations soluable in closed form. Further, because of rapid relative decay of the modified function with increasing order, five or six terms should be adequate for engineering accuracy.

For non-tabulated arguments the modified Bessel function of the first kind,

$$I_m\ (s) = i^{-m}\ J_m\ (is),$$

where $J_m\ (s)$ is the Bessel function of the first kind is defined by:

$$I_m\ (s) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+m}}{k\ !\ (k+m)!} \tag{B-6}$$

$$= \frac{s^m}{2^m\ \Gamma\ (m+1)} \left(1 + \frac{s^2}{2(2m+2)} + \frac{s^4}{2.4(2m+2)(2m+4)} + \ldots\right) \qquad (\ |\mathrm{ARG}\ s| < \pi)$$

with $\Gamma\ (n)$ gamma function $= (n-1)! = (n-1)\ \Gamma\ (n-1)$.

For small values of s, (B-6) can be approximated by the leading terms:

$$I_m (s) \doteq \frac{1}{2^m\, m!}\, s^m \quad . \tag{B-7}$$

For large values of s,

$$I_m (s) \doteq \frac{e^s}{\sqrt{2\pi\, s}} \quad . \tag{B-8}$$

In manipulating the modified Bessel functions, the following recurrence relations are useful,

$$\frac{d}{ds} I_m (a\, s) = a\, I_{m-1} (a\, s) - \frac{m}{s} I_m (a\, s) \tag{B-9}$$

$$\frac{d}{ds} I_m (a\, s) = a\, I_{m+1} (a\, s) + \frac{m}{s} I_m (a\, s) \tag{B-10}$$

$$2 \frac{d}{ds} I_m (a\, s) = a \left[I_{m-1} (a\, s) + I_{m+1} (a\, s) \right] \tag{B-11}$$

$$I_{m-1} (a\, s) - I_{m+1} (a\, s) = \frac{2\, m}{a\, s} I_m (a\, s) \; . \tag{B-12}$$

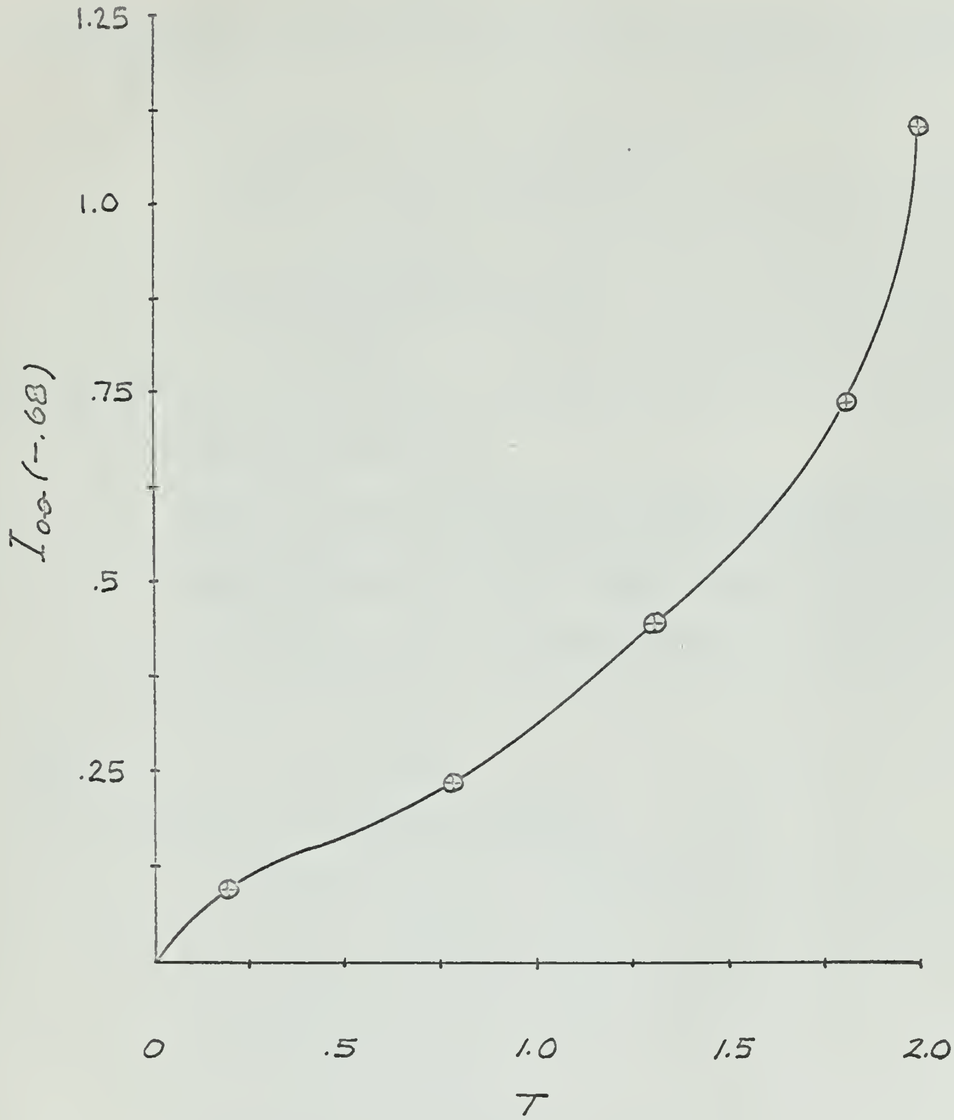


FIGURE B-1.
EXAMPLE PLOT OF INCOMPLETE MODIFIED BESSEL FUNCTION vs T (0< θ >2π)
n=0   S= -.680

APPENDIX C.

FOUR-POLE PARAMETERS

1. Development of Four-Pole Parameters for Fundamental Elements.

a. Mass

Performance Equations are:

$$\bar{V} = \bar{V}_1 = \bar{V}_2$$

$$\bar{F}_1 - \bar{F}_2 = s\, m\, \bar{V}$$

where F and V are arbitrary complex functions of time and $\bar{F}$ and $\bar{V}$ are the transforms of these functions. It is clear that the four-pole matrix is then:

$$\{m\} = \begin{bmatrix} 1 & sm \\ 0 & 1 \end{bmatrix}$$

b. Massless Spring Four Poles

Performance Equations are:

$$F = F_1 = F_2$$

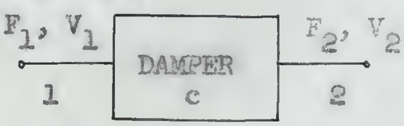
$$F_1 = k\left[x_1 - x_2\right] = \frac{k}{s}(V_1 - V_2),$$

$$\text{or} \quad V_1 = \frac{s}{k} F_2 + V_2$$

The four-pole matrix is then

$$\{k\} = \begin{bmatrix} 1 & 0 \\ s/k & 1 \end{bmatrix}$$

c. Viscous Damper



Performance Equations are:

$$F = F_1 = F_2,$$

$$F_1 = c(V_1 - V_2) \text{ or}$$

$$V_1 = \frac{1}{c} F_2 + V_2 .$$

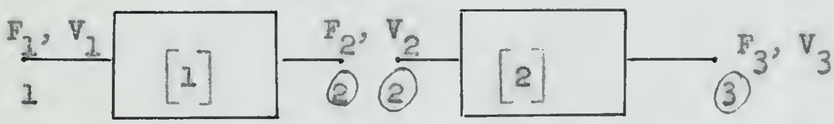
The four-pole matrix is thus

$$\{c\} = \begin{bmatrix} 1 & 0 \\ 1/c & 1 \end{bmatrix}$$

2. Combinations of Component Quadripoles

a. Series Combination.

If the output of one component is connected to the input station of another system, the two systems are in series. For example, if the two components below are



connected at point 2, the components are connected in series. If the four-pole of {1} is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and}$$

the four-pole of {2} is

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix}, \text{ then the}$$

four-pole relating F_1 and V_1 to F_3 and V_3 is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} .$$

So that the net series connected system four-pole is equal to the matrix product of the four-pole matrices associated with each of the systems making up the composite system.

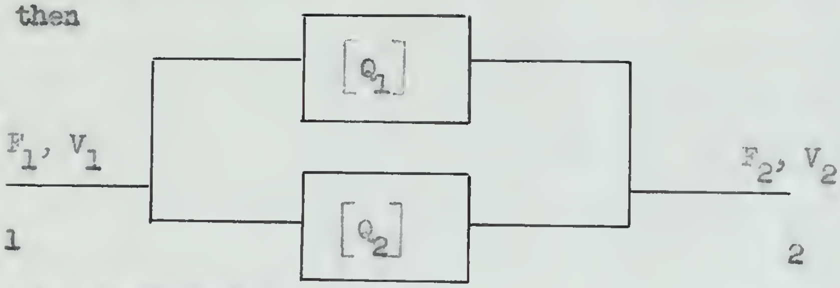
b. Parallel Combination.

Components are considered to be connected in parallel when:

(1) all the input and output junctions move with the same velocity

(2) the net input/output force of the resolvent quadripole is equal to the sum of the input/output forces of the constituent four poles.

If the system obeys Maxwell's Reciprocity Principle, then



can be replaced by

The matrix s of the composite system relates input to output by

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} s \end{bmatrix} \begin{bmatrix} F_2 \\ V_2 \end{bmatrix}$$

where $\begin{bmatrix} s \end{bmatrix} = \begin{bmatrix} D/B & -1/B \\ 1/B & -A/B \end{bmatrix}$

$$= \begin{vmatrix} D_1/B_1 & -1/B_1 \\ 1/B_1 & -A_1/B_1 \end{vmatrix} + \begin{vmatrix} D_2/B_2 & -1/B_2 \\ 1/B_1 & -A_2/B_2 \end{vmatrix}.$$

Application of the above formulas to a spring-damper parallel combination yield an equivalent four-pole

$$\begin{vmatrix} 1 & 0 \\ \frac{s}{rs+k} & 1 \end{vmatrix}.$$

More extensive developments are presented in (23), (1), and (24).

APPENDIX D.

IN-VACUO NORMAL MODES OF RING-STIFFENED SHELL

Analytic methods are available providing an additional measure of realism in the description of the modes and frequencies of in-vacuo ring-stiffened cylindrical shell. Because of the simple extension required to apply these frequencies to the quasi-impedance developed, the general form of an approximate method by Bleich[45] as reported in [43] are included here. Results of the approximate engineering analysis of [45] yield results in agreement with model experiments within $\pm$ 10 percent.

Nomenclature

ω_s = frequency of shell

ω_I = frequency of ring

m_s = mass of shell per unit of area

m_I = mass of ring frame per unit length

L_F = frame spacing

I = second moment of inertia of ring frame section plus a length of shell plating equal to $2/\rho$.

K & $\bar{\bar{K}}$ = Bleich constants tabulated in (46)

b = thickness of shell

ν = Poisson's ratio

E = Young's Modulus

a = mean shell radius

Bleich represents the total frequency of the in-vacuo vibration of stiffened cylinders as a summation of the shell frequencies plus the ring frequencies,

$$\omega^2 = \omega_s^{\,2} + \omega_I^{\,2} \tag{D-1}$$

$$\omega_I^{\,2} = n^2 \frac{(n^2 - 1)^2}{n^2 + 1} \times \frac{1}{a^4} \times \frac{EI}{m_r} \tag{D-2}$$

$$\omega_s^{\,2} = \frac{K^2}{a^2} \times \frac{Eb}{2(1-\upsilon)\,\bar{m}_s} \times \left[1 + \frac{\bar{K}^2}{K^2} \times \frac{b^2}{a^2}\right] \tag{D-3}$$

where $m_r = \bar{\alpha}_I \times L_F \times \bar{m}_s$,

$$\bar{m}_s = \frac{\alpha_I}{\bar{\alpha}_I} \times m_s + \frac{m_I}{\bar{\alpha}_I \cdot L_F} ,$$

$$\bar{\alpha}_I = \left[1 + \frac{e}{a}(n^2 - 1)\right]^2,$$

$$\alpha_I = \frac{2\int_0^{L_F/2} \left[\{1 + \frac{e}{a}(n^2 - 1) - \frac{e}{a}(n^2 - 1)\,\Phi(x)\}^2 + \frac{1}{n^2}\right] dx}{L_F\,(1 + \frac{1}{n^2})} ,$$

$\Phi(x) = e^{-\zeta x}\left[\cos \zeta x + \sin \zeta x\right]$, and

$$\zeta = \sqrt[4]{\frac{3(1 - \upsilon^2)}{a^2\, b^2}}$$

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